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Bounding the size of permutation groups and complex linear groups of odd order

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ABSTRACT

We provide improved bounds for the order of finite odd order permutation groups and complex linear groups, expressed in terms of the smallest prime divisor of the group order.

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Introduction

Numerous results exist in the literature bounding the orders of finite odd order subgroups of permutation groups and complex linear groups (see, for example, Gow [2], where such bounds were used to bound the number of characters in a p -block of a finite group). Our aim here is to point out that these bounds can be substantially improved if we place restrictions on the smallest prime divisor of the group order.

Given an odd prime p , we define constants $\alpha(p)$ and $\beta(p)$ as follows:

- i) $\alpha(3) = \sqrt{3}$, $\alpha(5) = 5^{\frac{1}{4}}$ and $\alpha(p) = [p(2p+1)]^{\frac{1}{2p}}$ for $p \geq 7$.
- ii) $\beta(p) = [p(2p+1)^2]^{\frac{1}{2p}}$ if $p \equiv 2 \pmod{3}$ and $\beta(p) = [p(2p-1)^2]^{\frac{1}{2p-2}}$ otherwise.

We will prove:

Theorem.

- a) Let G be a subgroup of odd order of the symmetric group S_n . Then $|G| \leq \alpha(p)^{n-1}$, where p is the smallest prime divisor of $|G|$.

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- b) Let G be a finite subgroup of odd order of $\text{GL}(n, \mathbb{C})$ and let p be the smallest prime divisor of $|G|$. Then:
- $[G : F(G)] \leq \alpha(p)^{n-1}$ and
 - G has an Abelian normal subgroup A with $[G : A] \leq \beta(p)^{n-1}$.

Remark. Notice that $\beta(3) = 75^{\frac{1}{4}} < 3$, $\alpha(5) < 1.5$ and $\beta(5) = 605^{\frac{1}{10}} < 1.9$. Also, $\alpha(p) = \max\{p^{\frac{1}{p-1}}, [p(2p+1)]^{\frac{1}{2p}}\}$ for all odd primes p . For when $p = 3$, we have $\sqrt{3} > 21^{\frac{1}{6}}$, when $p = 5$, we have $5^{\frac{1}{4}} > 55^{\frac{1}{10}}$, and when $p \geq 7$, we have $2^{p-1} > p^2$, so that $[p(2p+1)]^{p-1} > p^{2p}$. We will see later that $\alpha(p) < p^{\frac{3}{2(p-1)}}$ and $\beta(p) < p^{\frac{5}{2(p-1)}}$ for all odd primes p .

We also remark that when $n = p^k$ for some prime p and positive integer k , a Sylow p -subgroup of S_n has order $p^{\frac{n-1}{p-1}}$.

The theorem above has some consequences for the structure of general finite (solvable) groups of odd order.

For a finite group G , let $F(G)$ denote the Fitting subgroup of G and $\Phi(G)$ denote the Frattini subgroup of G . Let $F_k(G)$ denote the k -th Fitting subgroup of G , which is defined inductively via: $F_1(G) = F(G)$, $F_k(G)/F_{k-1}(G) = F(G/F_{k-1}(G))$.

Corollary. Let G be a finite solvable group of odd order and let p be the smallest prime divisor of $|G|$. Then:

i)

$$[G : F_2(G)] < [F(G) : \Phi(G)]^{\frac{1}{p-1}}$$

if $p \in \{3, 5\}$ and

$$[G : F_2(G)] < [F(G) : \Phi(G)]^{\left(\frac{1}{p} + \frac{\log 2}{2p \log p} + \frac{1}{4p^2 \log p}\right)}$$

otherwise.

- ii) $[G : \Phi(G)] < \frac{[F(G) : \Phi(G)]^{\frac{2p-1}{p-1}}}{2}$ if $p \in \{3, 5\}$ and $[G : \Phi(G)] < \frac{[F(G) : \Phi(G)]^{\frac{4p-1}{2p-2}}}{2}$ otherwise.

1. Preparatory results

1.1. General principles

When attempting to bound the size of subgroups of S_n , it is common to seek a bound of the form $|G| < c^{n-1}$ for some constant c if G belongs to a chosen collection \mathcal{C} of finite groups (we remark in passing that there is no such constant c when \mathcal{C} contains all finite groups, since $n!$ grows faster than c^n for any constant c). However, such bounds occur frequently: if G is a nilpotent subgroup of S_n , then it is well known that $|G| \leq 2^{n-1}$, and equality is attained whenever $n = 2^k$, taking G as a Sylow 2-subgroup of S_n . The constant $c = 24^{\frac{1}{3}}$ is known to be minimal choice of c when \mathcal{C} is the collection of all finite solvable groups.

Such bounds are amenable to inductive proofs as we now indicate. If \mathcal{C} is closed under taking normal subgroups and homomorphic images, and we know that the bound holds for groups in \mathcal{C} of order less than $|G|$, then we may reduce to the primitive case as follows. If G is intransitive, but has r orbits of respective sizes n_1, n_2, \dots, n_r , then G embeds in a direct product $G_1 \times \dots \times G_r$, where G_i is a homomorphic image of G and is also a transitive subgroup of S_{n_i} , so $|G| \leq \prod_{i=1}^r |G_i| \leq \prod_{i=1}^r c^{n_i-1}$. Hence we may assume that G is transitive. If G is transitive, but imprimitive, let H be a point stabiliser, and let K be a subgroup with $H \max K < G$, say $[G : K] = a$ and $b = [K : H]$, so that $n = ab$. Let N be the intersection of the G -conjugates of K . Then $N \triangleleft G$ and N is not contained in H ,

so that $K = HN$. By induction, we may suppose $[G : N] \leq c^{a-1}$. Now $[N : N \cap H] = b$ so N has an orbit of size b on the cosets of H . Thus all orbits of N on the cosets of H have size b and $|N| \leq (c^{b-1})^a$. Then $|G| \leq c^{ab-1}$. So it suffices to consider the case that G is primitive.

If, in addition, \mathcal{C} is closed under taking arbitrary subgroups, we can reduce the proof that if $G \in \mathcal{C}$ is a subgroup of $\text{GL}(n, \mathbb{C})$ for some n , then G has an Abelian normal subgroup A of index at most d^{n-1} , where d is a constant greater than or equal to c above, to the case that G is a primitive linear group. If the given representation of G is induced from a primitive representation of degree b of a proper subgroup H of index m , then $[G : N] \leq c^{m-1}$, where $N = \bigcap_{g \in G} H^g$. Now by Clifford's theorem, N embeds in a direct product $N_1 \times N_2 \times \cdots \times N_m$, where N_i is a subgroup of $\text{GL}(b, \mathbb{C})$ and is a homomorphic image of N . For at least one value of i , there is an irreducible overgroup H_i of N_i which is the image of H under a primitive representation τ_i of degree b . Since H_i acts primitively, all its Abelian normal subgroups are central. Hence $[H_i : Z(H_i)] \leq d^{b-1}$. Thus $[N_i : Z(N_i)] \leq d^{b-1}$ and $[N : Z(N)] \leq \prod_{i=1}^m [N_i : Z(N_i)] \leq d^{m(b-1)}$. Hence $[G : Z(N)] \leq d^{mb-m} c^{m-1} \leq d^{mb-1}$. It is also possible in the linear case to work with a bound of the form d^{rn-s} for constants r and s when $d^s \leq c$, such as is done in Dornhoff [1] for finite solvable complex linear groups, but we restrict our discussion to the case $r = s = 1$ for ease of exposition.

A similar argument works to reduce the proof that $[G : F(G)] \leq t^{n-1}$ for a constant $t \geq c$ when $G \in \mathcal{C}$ is a finite subgroup of $\text{GL}(n, \mathbb{C})$. We do not repeat the details, save to remark that it is not necessary to assume that \mathcal{C} is closed under taking arbitrary subgroups. Closure under taking normal subgroups (and homomorphic images) will do: if we have bounded the $[N_i : F(N_i)]$ in this way, for the N_i as above, then we bound $[G : F(N)]$, so also bound $[G : F(G)]$. In the Abelian case, we needed the overgroup H_i of N_i to conclude that the particular choice of Abelian normal subgroup $Z(N_i)$ could be used. We had used an arbitrary Abelian normal subgroup A_i of N_i , the resulting Abelian normal subgroup of N need not have been normal in G .

In this paper, we wish to vary this type of argument slightly. We wish to allow the constants c , d and t to be replaced by functions which may depend on the group $G \in \mathcal{C}$. Examination of the arguments shows that this is permissible if the following conditions are satisfied (when \mathcal{C} is closed under taking subgroups and homomorphic images).

- i) $\max\{c(X), t(X)\} \leq d(X)$ for all $X \in \mathcal{C}$.
- ii) $c(X) \geq c(Y)$ whenever $X \in \mathcal{C}$ and Y is a section of X .
- iii) $d(X) \geq d(Y)$ whenever $X \in \mathcal{C}$ and Y is a section of X .
- iv) $t(X) \geq t(Y)$ whenever $X \in \mathcal{C}$ and Y is a section of X .

The arguments above show that when these conditions are satisfied, if we wish to prove that whenever $G \in \mathcal{C}$ is a subgroup of S_n , we have $|G| \leq c(G)^{n-1}$, it suffices to consider the case that G is primitive. Similarly, if we have established the bound for subgroups of S_n in \mathcal{C} , and we wish to prove that $[G : F(G)] \leq t(G)^{n-1}$ and that there is an Abelian normal subgroup A of G with $[G : A] \leq d(G)^{n-1}$, when $G \in \mathcal{C}$ is a finite subgroup of $\text{GL}(n, \mathbb{C})$, then it suffices to consider the case that G is (irreducible and) primitive as a linear group.

We will use the functions $\alpha(p)$ in the role of both $d(G)$ and $t(G)$, and $\beta(p)$ in the role of $c(G)$, where p is the smallest prime divisor of $|G|$. The collection \mathcal{C} we are considering is the collection of finite (solvable) groups of odd order. If Y is a section of $X \in \mathcal{C}$, then the smallest prime divisor of $|Y|$ is certainly greater than or equal to the smallest prime divisor of $|X|$. We need to show, then, that functions α and β decrease as primes increase, and that we have $\alpha(p) \leq \beta(p)$ for all odd primes p .

We need some elementary calculus. We define functions f , g , h , k on (e, ∞) , by $f(x) = x^{\frac{1}{x-1}}$, $g(x) = [x(2x+1)]^{\frac{1}{2x}}$, $h(x) = [x(2x-1)^2]^{\frac{1}{2x-2}}$ and $k(x) = [x(2x+1)^2]^{\frac{1}{2x}}$.

Lemma 1. *The real valued functions f , g , h , k are strictly decreasing on their domains. Furthermore, $h(x) \geq k(x) \geq \max\{f(x), g(x)\}$ for all $x \geq 3$.*

Proof. The monotonicity is an exercise in elementary calculus. We next notice that $h(x) \geq h(x+1) > k(x)$ for $x > e$, and the last inequality is clear since $x^{\frac{3}{2x}} \geq x^{\frac{1}{x-1}}$ for $x \geq 3$. \square

Lemma 2. We have $k(x) > h(x+2)$ for $x \geq 4$.

Proof. We have $k(x) > (4x^3)^{\frac{1}{2x}}$ and

$$h(x+2) = [4x^3]^{\frac{1}{2x+2}} \left[\left(1 + \frac{2}{x}\right) \left(1 + \frac{3}{2x}\right)^2 \right]^{\frac{1}{2x+2}}.$$

Since $(1 + \frac{a}{x})^x \leq e^a$ for $a, x > 0$, we have

$$h(x+2) \leq [4x^3]^{\frac{1}{2x+2}} e^{\frac{5}{2(x^2+x)}}.$$

Now $\frac{k(x)}{h(x+2)} \geq (4x^3 e^{-5})^{\frac{1}{2(x^2+x)}}$. When $x \geq 4$, we have $4x^3 \geq 256 > 243 = 3^5 > e^5$, so the result follows. \square

Corollary 3. Let p and q be odd primes with $p < q$.

- i) $\alpha(p) > \alpha(q)$ and $\beta(p) > \beta(q)$.
- ii) If $\frac{q+1}{2}$ is odd and greater than p , then we have $\beta(p)^{q-1} \geq q^2(\frac{q+1}{2})$.
- iii) $\alpha(q) \leq q^{\frac{1}{4}}$ for all primes $q > 3$.

Proof. The first claim of i) follows because of the monotonicity of f and g . For the second one, if $\beta(p) = h(p)$, then we have $\beta(p) = h(p) > h(q) \geq k(q)$. If $\beta(p) = k(p)$, then $q \geq p+2$ and $p \equiv 2 \pmod{3}$, so that $p \geq 5$. Then $\beta(p) = k(p) > h(p+2) \geq h(q) > k(q)$.

For ii), note that $h(\frac{q+1}{2})^{q-1} = q^2(\frac{q+1}{2})$. Also, in this case $\frac{q+1}{2} \geq (p+2)$. It suffices to prove that $\beta(p) \geq h(\frac{q+1}{2})$. If $\beta(p) = h(p)$, this is clear. If not, then $p \equiv 2 \pmod{3}$ so $p \geq 5$ and $\beta(p) = k(p) > h(p+2) \geq h(\frac{q+1}{2})$.

iii) We may suppose that $q \geq 7$. Then

$$\alpha(q) = [q(2q+1)]^{\frac{1}{2q}} = q^{\frac{1}{2q}}(2q+1)^{\frac{1}{2q}} < q^{\frac{1}{2(q-1)}} q^{\frac{1}{q-1}} = q^{\frac{3}{2(q-1)}}. \quad \square$$

2. Some assumed results

We summarise some well-known facts about representations of solvable groups which we will freely use. We outline a proof in some cases. If G is a finite solvable group, then $F(G/\Phi(G)) = F(G)/\Phi(G)$ and this is a completely reducible module for $G/F(G)$. For each prime divisor of $|F(G)|$, $O_p(G)/O_p(\Phi(G))$ is a completely reducible $\text{GF}(p)G$ -module, on which $O_p(G)C_G(O_p(G))$ acts trivially.

If q is odd, then an Abelian $\{2, q\}'$ -subgroup of $\text{Sp}(2m, q)$ has order at most $\frac{q^m+1}{2}$. If q is odd, then an Abelian $\{2, q\}'$ -subgroup of $\text{GL}(m, q)$ has order at most $\frac{q^m-1}{2}$.

If G is a nilpotent $\{2, q\}'$ -subgroup of $\text{GL}(n, q)$ where q is an odd prime, then $|G| \leq \frac{q^n-1}{2}$. We may suppose that G is irreducible. If G is imprimitive, and the given representation is induced from a subgroup H of prime index r , then $H \triangleleft G$ as G is nilpotent. By induction, we may suppose that $|G| \leq r(\frac{q^b-1}{2})^r \leq \frac{q^b-1}{2}$. If G is primitive, then G is cyclic, and $|G| \leq \frac{q^n-1}{2}$, as claimed.

In particular, if G is a completely reducible nilpotent odd order subgroup of $\text{GL}(n, q)$ for an odd prime q , then $|G| \leq \frac{q^n-1}{2}$.

Returning to the completely reducible action of $G/F(G)$ on $F(G)/\Phi(G)$ we have $[F_2(G) : F(G)] \leq \frac{|F(G)\Phi(G)|-1}{2}$ when G is solvable of odd order. For each prime divisor q of $|F(G)|$, we have a homomorphism $\phi_q : G/F(G) \rightarrow \text{Aut}(O_q(G)/O_q(\Phi(G)))$, and the image of ϕ_q is a completely reducible

subgroup of $\mathrm{GL}(m, q)$ where $q^m = [O_q(G) : O_q(\Phi(G))]$. The image of $F_2(G)/F(G)$ is a completely reducible nilpotent subgroup of odd order of $\mathrm{GL}(m, q)$, so this image has order at most $\frac{q^m-1}{2}$. Since $G/F(G)$ embeds in the direct product $\prod_{q \in \pi(F(G))} \mathrm{Im} \phi_q$, the stated bound follows.

We will make implicit use of the fact that if G is a finite group, then $|G|$ and $|G/\Phi(G)|$ have the same prime divisors. For if some Sylow q -subgroup Q of G is contained in $\Phi(G)$, then $Q \triangleleft G$ and there is a complement N to Q by the Schur–Zassenhaus theorem, a contradiction.

When q is an odd prime, a completely reducible subgroup G of odd order of $\mathrm{Sp}(2m, q)$ lifts to a finite subgroup of $\mathrm{GL}(m, \mathbb{C})$. To see this, we may suppose that G is irreducible. Then, after extending the ground field to a splitting field, the natural module decomposes as a sum of r Galois conjugate absolutely irreducible modules of equal dimension d , each of which affords a faithful representation of G . By the Fong–Swan theorem, G is isomorphic to an irreducible subgroup of $\mathrm{GL}(d, \mathbb{C})$. Hence d must be odd, so that r is even and $d \leq m$.

Let G be a finite primitive solvable subgroup of odd order of $\mathrm{GL}(n, \mathbb{C})$ where $n > 1$. Suppose that $p_1 < p_2 < \dots < p_r$ are the prime divisors of $[F(G) : Z(G)]$, and set $P_i = \Omega_1(O_{p_i}(G))$ for each i . Then each P_i has nilpotence class 2 and is extra-special of exponent p_i , $Z(G)$ is cyclic, and $F(G) = Z(G)P_1 \dots P_r$. Also $P_i/Z(P_i)$ is a completely reducible module for G (and $O_{p_i}(G)C_G(O_{p_i}(G))$ acts trivially on it), since whenever $Q_i \triangleleft G$ with $Q_i \leq P_i$, we have $P_i = Q_i C_{P_i}(Q_i)$.

Let $|P_i| = p_i^{2d_i+1}$ for each i . Then $\prod_{i=1}^r p_i^{d_i}$ divides n and for each i , there is a homomorphism $\phi_i : G \rightarrow \mathrm{Sp}(2d_i, p_i)$. Let G_i denote the image of G and K_i denote the kernel of ϕ_i . Then $\bigcap_{i=1}^r K_i = F(G)$.

Hence $G/F(G)$ is isomorphic to a subgroup of the direct product $G_1 \times \dots \times G_r$. Letting V_i denote the natural module for $\mathrm{Sp}(2d_i, p_i)$, we note also that $G/Z(G)$ is isomorphic to a subgroup of $(V_1 G_1) \times \dots \times (V_r G_r)$. Each G_i is a completely reducible odd order subgroup of $\mathrm{Sp}(2d_i, p_i)$, so lifts to a finite subgroup of $\mathrm{GL}(d_i, \mathbb{C})$. By Clifford's theorem, $F(G_i)$ is also completely reducible, so, in particular, $F(G_i)$ is a p_i' -group. We have $[G : F(G)] \leq \prod_{i=1}^r |G_i|$ and $[G : Z(G)] \leq \prod_{i=1}^r p_i^{2d_i} |G_i|$. We remark here that $d_i > 1$ if p_i is the smallest prime divisor of $|G|$, since otherwise $i = 1$ and G_1 is isomorphic to an odd order subgroup of $\mathrm{SL}(2, p_1)$. Hence $F(G_1)$ is a p_1' -group, so that $|F(G_1)|$ divides $\frac{(p_1-1)(p_1+1)}{4}$, contrary to the assumption that p_1 is the smallest prime divisor of $|G|$.

Also, if $d_i = 2$ for any i , then G_i lifts to an odd order finite subgroup of $\mathrm{GL}(2, \mathbb{C})$, so G_i is Abelian (hence G_i is a p_i' -prime group, since $O_{p_i}(G_i) = 1$).

We also remark that the symplectic action of G_i allows us to construct the semi-direct product $P_i G_i$ with G_i acting trivially on $Z(P_i)$. The complex irreducible character of degree $p_i^{d_i}$ of P_i is uniquely determined by the linear character of $Z(P_i)$ it covers, and extends to $P_i \mathrm{Sp}(2d_i, p_i)$, so in particular to $P_i G_i$. (The extension to $\mathrm{Sp}(2d_i, p_i)$ is uniquely specified on p -regular elements if we insist (as we may) that they act with determinant 1. This determines the extension uniquely unless $d_i = 1$ and $p_i = 3$. But in the exceptional case, the unique unimodular extension to the normal subgroup $P_i Q_8$ is $\mathrm{SL}(2, p_i)$ -stable, so extends to $P_i \mathrm{SL}(2, p_i)$ in three ways as $P_i Q_8$ is a normal subgroup of prime index.)

Let A_i be an Abelian normal subgroup of G_i of maximal order (this need not be unique). Then we have $|A_i| \leq \frac{p_i^{d_i+1}}{2}$ and $|F(G_i)| \leq \frac{p_i^{2d_i}-1}{2}$ for each i .

3. Proof of the theorem

Proof. We prove a) and b) together by induction. If $|G| = p$, the result is clear. Suppose that $|G| > p$, and that a) and b) have been established for groups of order less than $|G|$. Notice that for every odd prime p , we have $\beta(p) \leq \beta(3) = 75^{\frac{1}{4}} < 3$.

To prove a), it suffices to treat the case that G is a (transitive and) primitive permutation group, so suppose this is the case. Then $G = VM$, where V is elementary Abelian of order $n = q^s$ and M is an irreducible odd order subgroup of $\mathrm{GL}(s, q)$. Then M lifts to a subgroup of $\mathrm{GL}(s, \mathbb{C})$. By the inductive hypothesis, M has an Abelian normal subgroup A of index at most $\beta(p)^{s-1} < 3^{s-1}$. Now $A \triangleleft M$ and M acts faithfully and irreducibly on V , so that A acts completely reducibly on V by

Clifford's theorem. Hence A is a q' -group, so that $|A| \leq \frac{q^s-1}{2}$. Then $|G| \leq q^s 3^{s-1} \frac{q^s-1}{2}$. Then certainly $|G| < n^3$.

If $s \geq 3$, then $1 + q + \dots + q^{s-1} \geq sq^{\frac{s-1}{2}} \geq 3s$ by the arithmetic-geometric mean inequality. Then

$$\alpha(p)^{n-1} \geq \alpha(q)^{n-1} \geq q^{\frac{n-1}{q-1}} \geq q^{3s} = n^3.$$

Hence we may suppose that $s < 3$.

If $s = 2$, then M is isomorphic to a subgroup of $GL(2, \mathbb{C})$. Thus M is an Abelian q' -group, and $|G| < q^4$. Then certainly

$$|G| < q^{q+1} = q^{\frac{n-1}{q-1}} \leq p^{\frac{n-1}{p-1}} \leq \alpha(p)^{n-1}.$$

Hence we may suppose that $s = 1$.

Now $n = q$ for some prime q and $|G|$ divides $\frac{q(q-1)}{2}$. Then $p \leq \frac{q-1}{2}$ and

$$\alpha(p)^{q-1} \geq g(p)^{q-1} \geq g\left(\frac{q-1}{2}\right)^{q-1} = \frac{q(q-1)}{2}.$$

b) Now suppose that G is a subgroup (of odd order) of $GL(n, \mathbb{C})$, and parts a) and b) hold for groups of smaller (odd) order. We will prove that G has an Abelian normal subgroup of index at most $\beta(p)^{n-1}$ and that $[G : F(G)] \leq \alpha(p)^{n-1}$. Since $\alpha(p) \leq \beta(p)$ for all p , we may suppose that G is irreducible and primitive, and that $n > 1$, so we do.

Let $[F(G) : Z(G)] = p_1^{2d_1} \dots p_r^{2d_r}$ where $p_1 < \dots < p_r$ are primes. Then $\prod_{i=1}^r p_i^{d_i}$ divides n . Define the groups G_i and A_i as in the discussion of Section 2. Each G_i is isomorphic to a subgroup of $GL(d_i, \mathbb{C})$, so by the inductive hypothesis, we have $[G_i : F(G_i)] \leq \alpha(p)^{d_i-1}$ and $[G_i : A_i] \leq \beta(p)^{d_i-1}$ for each i .

Then

$$[G : Z(G)] \leq \prod_{i=1}^r p_i^{2d_i} |G_i| \leq \prod_{i=1}^r p_i^{2d_i} \left(\frac{p_i^{d_i} + 1}{2}\right) \beta(p)^{d_i-1}.$$

Also,

$$[G : F(G)] \leq \prod_{i=1}^r |G_i| \leq \prod_{i=1}^r \left(\frac{p_i^{2d_i} - 1}{2}\right) \alpha(p)^{d_i-1}.$$

It suffices to prove that for each i we have $p_i^{2d_i} |G_i| \leq \beta(p)^{p_i^{d_i}-1}$ and $|G_i| \leq \alpha(p)^{p_i^{d_i}-1}$. In other words, it suffices to establish the stated bounds for each of the groups $P_i G_i$, viewed as subgroups of $GL(p_i^{d_i}, \mathbb{C})$, since $P_i = F(P_i G_i)$ and $Z(P_i)$ is the largest Abelian normal subgroup of $P_i G_i$.

Hence we may suppose that $r = 1$, that $F(G)$ is a p_1 -group, that $G = P_1 G_1$, and that $n = p_1^{d_1}$. If we can prove that

$$p_1^{2d_1} \left(\frac{p_1^{d_1} + 1}{2}\right) \beta(p)^{d_1-1} \leq \beta(p)^{p_1^{d_1}-1},$$

for each i , we will have established part i) of b).

This is certainly true if $d_1[3 \log_{\beta(p)}(p_1) + 1] \leq p_1^{d_1}$. Now $\beta(p) \geq \alpha(p) \geq p_1^{\frac{1}{p_1-1}}$, so that $\log_{\beta(p)}(p_1) \leq p_1 - 1$. The required inequality holds if $d_1(3p_1 - 2) \leq p_1^{d_1}$. This holds without exception when $d_1 \geq 3$.

If $d_1 = 2$, then the required inequality holds for $p_1 > 5$. If $p_1 = 5$ and $d_1 = 2$, we return to the finer estimates. Then $p \in \{3, 5\}$, so that $\beta(p) \geq 5^{\frac{1}{4}}$. We need to show that $13 \times 5^4 \leq \beta(p)^{23}$. But $5^{\frac{23}{4}} > 3 \times 5^5$, and the necessary inequality holds. If $p_1 = 3$ and $d_1 = 2$, then $p_1^{2d_1} |G_1| = 405$. But $\beta(3) = 75^{\frac{1}{4}}$ and $p^{d_1} - 1 = 8$, so $p_1^{2d_1} |G_1| \leq \beta(p)^{n-1}$.

If $d_1 = 1$, then $p_1^{2d_1} |G_1| \leq \frac{p_1^2(p_1+1)}{2}$. In that case, p_1 is not the smallest prime divisor of $|G|$.

In this case, G_1 is a non-trivial Abelian p_1' -subgroup of $SL(2, p_1)$ of order dividing $\frac{p_1+\varepsilon}{2}$ for some sign ε . Now p is a divisor of $[G : O_{p_1}(G)] = |G_1|$.

If $p = \frac{p_1+1}{2}$, then $\beta(p)^{p_1-1} = h(p)^{p_1-1} = \frac{p_1^2(p_1+1)}{2}$ and $n = p_1$. If $p \leq \frac{p_1-1}{2}$ then

$$\beta(p)^{p_1-1} \geq k(p)^{p_1-1} \geq k\left(\frac{p_1-1}{2}\right)^{p_1-1} = \frac{p_1^2(p_1-1)}{2}.$$

Hence we may suppose that $[G : Z(G)] = p_1^2\left(\frac{p_1+1}{2}\right)$, and that p divides $\frac{p_1+1}{2}$, but $p \neq \frac{p_1+1}{2}$. Then $\frac{p_1+1}{2}$ is certainly odd, so that $\frac{p_1+1}{2} \geq p+2$. Now by Lemma 2, we have

$$\alpha(p)^{p_1-1} \geq k(p)^{p_1-1} > h(p+2)^{p_1-1} \geq h\left(\frac{p_1+1}{2}\right)^{p_1-1} = \frac{p_1^2(p_1+1)}{2},$$

so b) part i) is now proved.

To prove part ii) of b), we need to prove that

$$|G_1| \leq \alpha(p)^{p_1^{d_1-1}}.$$

We observed earlier in the proof (using the inductive hypotheses) that $|G_1| < \alpha(p)^{d_1-1}(p_1^{2d_1} - 1)$.

Since $\alpha(p) \leq \sqrt{p} \leq \sqrt{p_1}$, we obtain $|G_1| < p_1^{\frac{5d_1}{2}}$.

On the other hand $\alpha(p) \geq p^{\frac{1}{p-1}} \geq p_1^{\frac{1}{p_1-1}}$, so that

$$\alpha(p)^{p_1^{d_1-1}} \geq p_1^{\frac{p_1^{d_1-1}-1}{p_1-1}} \geq p_1^{d_1 p_1^{\frac{d_1-1}{2}}}.$$

Since $p_1 \geq 3$, we have $d_1 < 3$. If $d_1 = 2$, we can calculate more carefully. We have

$$\alpha(p)^{p_1^{d_1-1}} \geq p_1^{\frac{p_1^2-1}{p_1-1}} = p_1^{1+p_1}$$

and we know that $|G_1| < p_1^5$. Thus $p_1 = 3$. In that case G_1 is a completely reducible odd order subgroup of $Sp(4, 3)$. Thus $|F(G_1)| = 5$ and in fact $|G_1| = 5$. But $p_1^{1+p_1} = 81$ in this case. Hence $d_1 = 1$.

But now G_1 is an odd order completely reducible subgroup of $SL(2, p_1)$ so is Abelian of order at most $\frac{p_1+1}{2}$. However, $\alpha(p)^{p_1-1} \geq p_1$, so the proof is complete. \square

Proof of the corollary. $G/F(G)$ acts completely reducibly on $F(G)/\Phi(G)$, and the latter group is Abelian of squarefree exponent. Applying the Fong–Swan theorem to the linear action on each minimal normal subgroup, we see that $G/F(G)$ is isomorphic to a subgroup of $GL(n, \mathbb{C})$ where n is the number of prime factors (including multiplicities) of $[F(G) : \Phi(G)]$. Then certainly $n \leq \log_p([F(G) : \Phi(G)])$.

Applying b)ii) of the theorem to $G/F(G)$, we see that $[G : F_2(G)] < \alpha(p)^n$ so that

$$[G : F_2(G)] < \alpha(p)^{\log_p([F(G) : \Phi(G)])}.$$

When $p \in \{3, 5\}$, we have $\alpha(p) = p^{\frac{1}{p-1}}$. When $p > 5$, we have

$$\log(\alpha(p)) = \frac{\log p}{p} + \frac{\log 2}{2p} + \frac{\log(1 + \frac{1}{2p})}{2p} < \frac{\log p}{p} + \frac{\log 2}{2p} + \frac{1}{4p^2}.$$

Hence

$$\log([G : F_2(G)]) < \log([F(G) : \Phi(G)]) \left[\frac{1}{p} + \frac{\log 2}{2p \log p} + \frac{1}{4p^2 \log p} \right]$$

and the result follows.

iv) It suffices to consider the case $\Phi(G) = 1$. We have $[F_2(G) : F(G)] \leq \frac{|F(G)|-1}{2}$, while from part ii) we have $[G : F_2(G)] < \alpha(p)^{\log_p(|F(G)|)}$. We know that $\alpha(p) = p^{\frac{1}{p-1}}$ when $p \in \{3, 5\}$, while we have seen that $\alpha(p) < p^{\frac{3}{2(p-1)}}$ for $p > 5$. Hence we certainly have $[G : F(G)] < \frac{|F(G)|^{1+\frac{3}{2(p-1)}}}{2}$ for $p > 5$ and $[G : F(G)] < \frac{|F(G)|^{1+\frac{1}{p-1}}}{2}$ for $p \in \{3, 5\}$. \square

References

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